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Brans-Dicke theory and the emergence of ΛCDM model

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Abstract. Dynamics of the Brans-Dicke theory with a scalar field potential function is investigated. We show that the system with a barotropic matter content can be reduced to an autonomous three dimensional dynamical system. For an arbitrary potential function we found the values of the Brans-Dicke parameter for which a global attractor in the phase space representing deSitter state exists. Using linearised solutions in the vicinity of this critical point we show that the evolution of the universe mimics the ΛCDM model. From the recent Planck satellite data, we obtain constraint on the variability of the gravitational coupling constant as well as the lower limit of the mass of the Brans-Dicke scalar field at the deSitter state.

Keywords: modified gravity, dark energy theory, dark matter theory

Contents

1 Introduction

Recent astronomical observations [1] indicate that the Standard Cosmological Model (ΛCDM) is still a good effective theory of the physical Universe, although some anomalies in the power spectrum of CMB are predicted [2].

In this theory the crucial role play cosmological parameters which values are estimated from the astronomical observations. The nature of some parameters, like density parameters for dark sector of the Universe (dark matter and dark energy), is still unknown up to today and we are looking for an explanation. We believe that more fundamental theory of gravity will reveal the nature of the cosmological parameters. In other words we expect that the nature of these parameters is emergent.

The Brans-Dicke theory of gravity [3] seems to be an interesting way toward construction of some more adequate description of evolution of the Universe by which one can explain why the universe enters the accelerating phase of expansion just in the current epoch.

In this paper, we show how dynamical evolution of the standard cosmological model emerges from the cosmological model in which, instead of General Relativity, we use the Brans-Dicke theory of gravity. In principle, we obtain two different evolutional scenarios in which the ΛCDM model with some additional corrections is recovered. Those new additive terms appear in the equation describing the evolution of the Universe as well as the parameters describing the dark sector are modified. The generalised Friedmann-Robertson-Walker equation can be tested trough the astronomical observations and the model itself can be falsified be the data.

The gravitational sector of the Brans-Dicke theory is modified by the presence of some potential function, while the barotropic matter is used for description of the matter content of the universe. However, because of the indetermination of the form of the potential (for some new results concerning the scalar field potential function for the inflaton field see recent Planck results [4]) our strategy is to obtain results without the "potential bias" as soon as possible. Therefore in our analysis we concentrate on the behaviour of the dynamical system describing the evolution of the universe in the vicinity of a special critical point whose position in the phase space does not depend on the detailed form of the potential function.

We demonstrate that there are two different types of the emergence of the ΛCDM model. Different initial conditions and the model parameters give rise to different scenarios. In the first scenario the ΛCDM model is approached monotonically while in the second approach we obtain oscillatory approach to the deSitter state. In both scenarios, the leading term represents evolution in the ΛCDM model.

Only the astronomical data can help in selecting the type of evolution which is realised by our universe. Alternatively, deeper knowledge of the initial conditions and the model parameters, can distinguish a single type of the behaviour.

2 The Brans-Dicke cosmology

The action for the Brans-Dicke theory [3] in the so-called Jordan frame is in the following form [5, 6]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\phi R - \frac{\omega_{\text{BD}}}{\phi}\nabla^\alpha\phi\nabla_\alpha\phi - 2\,V(\phi)\right\} + 16\pi S_m \tag{2.1}$$

where the barotropic matter is described by

$$S_m = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m\,, \tag{2.2}$$

and ω_{BD} is a dimensionless parameter of the theory.

Variation of the total action (2.1) with respect to the metric tensor $\delta S/\delta g^{\mu\nu} = 0$ gives the field equations for the theory

$$\begin{aligned}&\phi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \\ &= \frac{\omega_{\text{BD}}}{\phi}\left(\nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}\,g_{\mu\nu}\,\nabla^\alpha\phi\,\nabla_\alpha\phi\right) - g_{\mu\nu}V(\phi) - (g_{\mu\nu}\Box\phi - \nabla_\mu\nabla_\nu\phi) + 8\pi\,T^{(m)}_{\mu\nu}\,,\end{aligned} \tag{2.3}$$

where the energy momentum tensor for the matter content is

$$T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-g}\mathcal{L}_m\right)\,. \tag{2.4}$$

Taking the trace of (2.3) one obtains

$$R = \frac{\omega_{\text{BD}}}{\phi^2}\,\nabla^\alpha\phi\,\nabla_\alpha\phi + 4\frac{V(\phi)}{\phi} + 3\frac{\Box\phi}{\phi} - \frac{8\pi}{\phi}T^m. \tag{2.5}$$

Variation of the action with respect to ϕ gives

$$\Box\phi = \frac{1}{2\phi}\nabla^\alpha\phi\,\nabla_\alpha\phi - \frac{\phi}{2\omega_{\text{BD}}}\left(R - 2V'(\phi)\right)\,, \tag{2.6}$$

and using (2.5) to eliminate the Ricci scalar R, one obtains

$$\Box\phi = -\frac{2}{3+2\omega_{\rm BD}}\left(2V(\phi)-\phi V'(\phi)\right) + \frac{8\pi}{3+2\omega_{\rm BD}}T^m\,. \tag{2.7}$$

The field equations (2.3) are adopted to the spatially flat Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\big(dx^2+dy^2+dz^2\big) \tag{2.8}$$

and the matter content is described by the barotropic equation of state $p_m = w_m\rho_m$, where p_m and ρ_m are the pressure and the energy density of the matter.

The energy conservation condition is

$$3H^2 = \frac{\omega_{\rm BD}}{2}\frac{\dot\phi^2}{\phi^2} + \frac{V(\phi)}{\phi} - 3H\frac{\dot\phi}{\phi} + \frac{8\pi}{\phi}\rho_m \tag{2.9}$$

and the acceleration equation is

$$\dot H = -\frac{\omega_{\rm BD}}{2}\frac{\dot\phi^2}{\phi^2} - \frac{1}{3+2\omega_{\rm BD}}\frac{2\,V(\phi)-\phi V'(\phi)}{\phi} + 2H\frac{\dot\phi}{\phi} - \frac{8\pi}{\phi}\rho_m\frac{2+\omega_{\rm BD}(1+w_m)}{3+2\omega_{\rm BD}}\,. \tag{2.10}$$

The dynamical equation (2.7) for the scalar field reduces to

$$\ddot\phi + 3H\dot\phi = 2\frac{2\,V(\phi)-\phi\,V'(\phi)}{3+2\omega_{\rm BD}} + 8\pi\rho_m\frac{1-3w_m}{3+2\omega_{\rm BD}}\,. \tag{2.11}$$

In what follows we introduce the following energy phase space variables

$$x \equiv \frac{\dot\phi}{H\phi}\,, \qquad y \equiv \sqrt{\frac{V(\phi)}{3\phi}}\frac{1}{H}\,, \qquad \lambda \equiv -\phi\frac{V'(\phi)}{V(\phi)}\,. \tag{2.12}$$

Then the energy conservation condition (2.9) can be presented as

$$\frac{8\pi\rho_m}{3H^2\phi} = 1 + x - \frac{\omega_{\rm BD}}{6}x^2 - y^2 \tag{2.13}$$

and the acceleration equation (2.10)

$$\begin{aligned}\frac{\dot H}{H^2} &= 2x - \frac{\omega_{\rm BD}}{2}x^2 - \frac{3}{3+2\omega_{\rm BD}}y^2(2+\lambda)\\ &\quad - 3\left(1+x-\frac{\omega_{\rm BD}}{6}x^2-y^2\right)\frac{2+\omega_{\rm BD}(1+w_m)}{3+2\omega_{\rm BD}}\,.\end{aligned} \tag{2.14}$$

The dynamical system describing the evolution of the Brans-Dicke theory of gravity with a scalar field potential and the barotropic matter has the following form

$$\begin{aligned}\frac{dx}{d\tau} &= -3x - x^2 - x\frac{\dot H}{H^2} + \frac{6}{3+2\omega_{\rm BD}}y^2(2+\lambda) + 3\left(1+x-\frac{\omega_{\rm BD}}{6}x^2-y^2\right)\frac{1-3w_m}{3+2\omega_{\rm BD}}\,,\\ \frac{dy}{d\tau} &= -y\left(\frac{1}{2}x(1+\lambda) + \frac{\dot H}{H^2}\right)\,,\\ \frac{d\lambda}{d\tau} &= x\lambda\Big(1-\lambda(\Gamma-1)\Big)\,,\end{aligned} \tag{2.15}$$

where $\frac{\mathrm{d}}{\mathrm{d}\tau}=\frac{\mathrm{d}}{\mathrm{d}\ln a}$ and

$$\Gamma=\frac{V''(\phi)V(\phi)}{V'(\phi)^2}\,, \tag{2.16}$$

where $()'=\frac{\mathrm{d}}{\mathrm{d}\phi}$.

From now on we will assume that $\Gamma=\Gamma(\lambda)$. The critical points of the system (2.15) depend on the explicit form of the $\Gamma(\lambda)$ function. One can notice that the critical points ($x^*=0\,,y^*=\pm1\,,\lambda^*=-2$) do not depend on the assumed $\Gamma(\lambda)$. Additionally, the acceleration (2.14) calculated at these points vanishes, giving rise to the deSitter evolution.

Our analysis we concentrate on the critical point ($x^*=0$, $y^*=1$, $\lambda^*=-2$) which corresponds to the deSitter state, while the critical point with $y^*=-1$ corresponds to the anti-deSitter state.

The linearisation matrix calculated at this point is

$$A=\begin{pmatrix} -3\frac{2+2\omega_{\rm BD}+3w_m}{3+2\omega_{\rm BD}} & -6\frac{1-3w_m}{3+2\omega_{\rm BD}} & \frac{6}{3+2\omega_{\rm BD}} \\ \frac{3}{2}\frac{1+2\omega_{\rm BD}w_m}{3+2\omega_{\rm BD}} & -6\frac{2+\omega_{\rm BD}(1+w_m)}{3+2\omega_{\rm BD}} & \frac{3}{3+2\omega_{\rm BD}} \\ \frac{3}{8}\delta & 0 & 0 \end{pmatrix} \tag{2.17}$$

where we assumed $\left.\frac{\partial\lambda'}{\partial x}\right|_*=\lambda^*\Big(1-\lambda^*(\Gamma(\lambda^*)-1)\Big)=\frac{3}{8}\delta$, and δ is a nonzero constant and $\left.\frac{\partial\Gamma(\lambda)}{\partial\lambda}\right|_*$ is finite. The eigenvalues are

$$l_1=-3(1+w_m)\,,\quad l_{2,3}=-\frac{3}{2}\left(1\pm\sqrt{\frac{3+2\omega_{\rm BD}+\delta}{3+2\omega_{\rm BD}}}\right)\,. \tag{2.18}$$

For $\delta=0$ one of the eigenvalues vanishes and this critical point is degenerated. Therefore the analysis based on the linearisation matrix is not enough and the center manifold theorem must be used [7–9].

The critical point under consideration is stable when $w_m>-1$ and

$$\frac{\delta}{3+2\omega_{\rm BD}}<0 \tag{2.19}$$

for $-1<\frac{\delta}{3+2\omega_{\rm BD}}<0$ is a stable node, while for $\frac{\delta}{3+2\omega_{\rm BD}}<-1$ is a stable focus. This indicates that for a given family of potential functions described by the function $\Gamma(\lambda)$ we can always find ranges of the $\omega_{\rm BD}$ parameter for which the critical point under considerations is a stable one. Let us consider two simple potential functions. For potential of the type $V(\phi)=V_0(\phi^2-v^2)^2$, we have $\delta=-\frac{16}{3}$, and for $\omega_{\rm BD}>\frac{7}{6}$ we have a stable node critical point, while for $-\frac{3}{2}<\omega_{\rm BD}<\frac{7}{6}$ we have a stable focus. It represents a saddle type critical point otherwise. For the potential type $V(\phi)=\frac{1}{2}m^2\phi^2+\frac{\alpha}{4}\phi^4$, we have $\delta=\frac{8}{3}$. For $\omega_{\rm BD}<-\frac{17}{6}$ this critical point represents a stable node, and for $-\frac{17}{6}<\omega_{\rm BD}<-\frac{3}{2}$ we have the critical point of a stable focus type.

We need to consider both types of behaviour separately.

In the first case we use the following substitution

$$\frac{\delta}{3+2\omega_{\rm BD}} = \frac{4}{9}n(n-3) \tag{2.20}$$

where for $0 < n < \frac{3}{2}$ we have a stable node critical point. The eigenvalues of the linearisation matrix are

$$l_1 = -3(1+w_m)\,, \quad l_2 = -n\,, \quad l_3 = -3+n\,, \tag{2.21}$$

and the linearised solutions are

$$\begin{aligned}
x(\tau) =\ & 4\frac{n(n-3)(1+w_m)(1-3w_m)}{\delta(n-3(1+w_m))(n+3w_m)}(\Delta x - 2\Delta y)\exp\left(-3(1+w_m)\tau\right) + \\
& + \frac{n}{3\delta(2n-3)(n-3(1+w_m))}\Big(-4n(n-3)(1-3w_m)(\Delta x - 2\Delta y) + \\
& \qquad + (n-3(1+w_m))(3\delta\Delta x - 8(n-3)\Delta\lambda)\Big)\exp\left(-n\tau\right) + \\
& + \frac{n-3}{3\delta(2n-3)(n+3w_m)}\Big(4n(n-3)(1-3w_m)(\Delta x - 2\Delta y) + \\
& \qquad + (n+3w_m)(3\delta\Delta x + 8n\Delta\lambda)\Big)\exp\big((-3+n)\tau\big)\,,
\end{aligned} \tag{2.22a}$$

$$\begin{aligned}
y(\tau) =\ & 1 + \left(2\frac{n(n-3)(1+w_m)(1-3w_m)}{\delta(n-3(1+w_m))(n+3w_m)} - \frac{1}{2}\right)(\Delta x - 2\Delta y)\exp\big(-3(1+w_m)\tau\big) + \\
& + \frac{n}{6\delta(2n-3)(n-3(1+w_m))}\Big(-4n(n-3)(1-3w_m)(\Delta x - 2\Delta y) + \\
& \qquad + (n-3(1+w_m))(3\delta\Delta x - 8(n-3)\Delta\lambda)\Big)\exp\big(-n\tau\big) + \\
& + \frac{n-3}{6\delta(2n-3)(n+3w_m)}\Big(4n(n-3)(1-3w_m)(\Delta x - 2\Delta y) + \\
& \qquad + (n+3w_m)(3\delta\Delta x + 8n\Delta\lambda)\Big)\exp\big((-3+n)\tau\big)\,,
\end{aligned} \tag{2.22b}$$

$$\begin{aligned}
\lambda(\tau) =\ & -2 - \frac{n(n-3)(1-3w_m)}{2(n-3(1+w_m))(n+3w_m)}(\Delta x - 2\Delta y)\exp\big(-3(1+w_m)\tau\big) + \\
& + \frac{1}{8(2n-3)(n-3(1+w_m))}\Big(4n(n-3)(1-3w_m)(\Delta x - 2\Delta y) - \\
& \qquad - (n-3(1+w_m))(3\delta\Delta x - 8(n-3)\Delta\lambda)\Big)\exp\big(-n\tau\big) + \\
& + \frac{1}{8(2n-3)(n+3w_m)}\Big(4n(n-3)(1-3w_m)(\Delta x - 2\Delta y) + \\
& \qquad + (n+3w_m)(3\delta\Delta x + 8n\Delta\lambda)\Big)\exp\big((-3+n)\tau\big)
\end{aligned} \tag{2.22c}$$

where $\Delta x = x^{(i)}$, $\Delta y = y^{(i)} - 1$ and $\Delta\lambda = \lambda^{(i)} + 2$ are the initial conditions.

Using these linearised solution and the conservation condition (2.13) we can find linearised solution for the barotropic matter density fraction. Up to linear terms in initial conditions we obtain

$$\Omega_m \approx (\Delta x - 2\Delta y) \exp\big(-3(1+w_m)\tau\big) = \Omega_{m,i} \exp\big(-3(1+w_m)\tau\big). \tag{2.23}$$

Now, using the linearised solutions, we are ready to compute the Hubble's function. The acceleration equation (2.14) can be cast into the following form

$$\frac{\mathrm{d}\ln H^2}{\mathrm{d}\tau} = 2\frac{\dot{H}}{H^2} = 4x - \omega_{\mathrm{BD}}x^2 - \frac{6}{3+2\omega_{\mathrm{BD}}}y^2(2+\lambda) - 6\left(1 + x - \frac{\omega_{\mathrm{BD}}}{6}x^2 - y^2\right)\frac{2+\omega_{\mathrm{BD}}(1+w_m)}{3+2\omega_{\mathrm{BD}}} \tag{2.24}$$

after integration of this equation, and up to the linear terms in initial conditions we obtain

$$\left(\frac{H(a)}{H(a_0)}\right)^2 \approx \Omega_{\Lambda,0} + \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3(1+w_m)} + \Omega_{n,0}\left(\frac{a}{a_0}\right)^{-n} + \Omega_{3n,0}\left(\frac{a}{a_0}\right)^{-3+n}, \tag{2.25}$$

where

$$\Omega_{M,0} = \left(1 - \frac{4n(n-3)(1-3w_m)(4+3w_m)}{3\delta(n+3w_m)(n-3(1+w_m))}\right)\Omega_{m,0}\,, \tag{2.26a}$$

$$\Omega_{n,0} = \frac{n+1}{3\delta(2n-3)}\left(\frac{4n(n-3)(1-3w_m)}{n-3(1+w_m)}\,\Omega_{m,i} - 3\delta\,\Delta x + 8(n-3)\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-n}, \tag{2.26b}$$

$$\Omega_{3n,0} = \frac{n-4}{3\delta(2n-3)}\left(-\frac{4n(n-3)(1-3w_m)}{n+3w_m}\,\Omega_{m,i} - 3\delta\,\Delta x - 8n\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-3+n}, \tag{2.26c}$$

and

$$\Omega_{\Lambda,0} = 1 - \Omega_{M,0} - \Omega_{n,0} - \Omega_{3n,0}\,, \tag{2.27}$$

and we have used that in linear approximation

$$\Omega_{m,0} = \Omega_{m,i}\left(\frac{a_0}{a^{(i)}}\right)^{-3(1+w_m)}. \tag{2.28}$$

Now we proceed to investigation of the critical point of the stable focus type. In this case we use the substitution

$$\frac{\delta}{3+2\omega_{\mathrm{BD}}} = -\frac{1}{9}(9+4n^2) \tag{2.29}$$

and the eigenvalues of the linearisation matrix are

$$l_1 = -3(1+w_m)\,, \quad l_2 = -\frac{3}{2} - \mathtt{i}n\,, \quad l_3 = -\frac{3}{2} + \mathtt{i}n\,. \tag{2.30}$$

The linearised solutions of the system are

$$
\begin{aligned}
x(\tau) = {} & \frac{4(4n^2+9)(1+w_m)(1-3w_m)}{\delta(4n^2+9(1+2w_m)^2)}\big(\Delta x - 2\Delta y\big)\exp\big(-3(1+w_m)\tau\big) + \\
& + \bigg(-\frac{4(4n^2+9)(1+w_m)(1-3w_m)}{\delta(4n^2+9(1+2w_m)^2)}\big(\Delta x - 2\Delta y\big) + \Delta x\bigg)\exp\big(-\frac{3}{2}\tau\big)\cos\big(n\tau\big) + \\
& + \frac{1}{6n}\bigg(-\frac{2(4n^2+9)(4n^2-9(1+2w_m))(1-3w_m)}{\delta(4n^2+9(1+2w_m)^2)}\big(\Delta x - 2\Delta y\big) - \\
& \qquad -9\Delta x - \frac{4}{\delta}(4n^2+9)\Delta\lambda\bigg)\exp\big(-\frac{3}{2}\tau\big)\sin\big(n\tau\big)\,,
\end{aligned} \tag{2.31a}
$$

$$
\begin{aligned}
y(\tau) = {} & 1 + \frac{1}{2}\bigg(\frac{4(4n^2+9)(1+w_m)(1-3w_m)}{\delta(4n^2+9(1+2w_m)^2)} - 1\bigg)\big(\Delta x - 2\Delta y\big)\exp\big(-3(1+w_m)\tau\big) + \\
& + \frac{1}{2}\bigg(-\frac{4(4n^2+9)(1+w_m)(1-3w_m)}{\delta(4n^2+9(1+2w_m)^2)}\big(\Delta x - 2\Delta y\big) + \Delta x\bigg)\exp\big(-\frac{3}{2}\tau\big)\cos\big(n\tau\big) + \\
& + \frac{1}{12n}\bigg(-\frac{2(4n^2+9)(4n^2-9(1+2w_m))(1-3w_m)}{\delta(4n^2+9(1+2w_m)^2)}\big(\Delta x - 2\Delta y\big) - \\
& \qquad -9\Delta x - \frac{4}{\delta}(4n^2+9)\Delta\lambda\bigg)\exp\big(-\frac{3}{2}\tau\big)\sin\big(n\tau\big)\,,
\end{aligned} \tag{2.31b}
$$

$$
\begin{aligned}
\lambda(\tau) = {} & -2 - \frac{(4n^2+9)(1-3w_m)}{2(4n^2+9(1+2w_m)^2)}\big(\Delta x - 2\Delta y\big)\exp\big(-3(1+w_m)\tau\big) + \\
& + \bigg(\frac{(4n^2+9)(1-3w_m)}{2(4n^2+9(1+2w_m)^2)}\big(\Delta x - 2\Delta y\big) + \Delta\lambda\bigg)\exp\big(-\frac{3}{2}\tau\big)\cos\big(n\tau\big) + \\
& + \frac{3}{8n}\bigg(-\frac{2(4n^2+9)(1+2w_m)(1-3w_m)}{4n^2+9(1+2w_m)^2}\big(\Delta x - 2\Delta y\big) + \\
& \qquad +\delta\Delta x + 4\Delta\lambda\bigg)\exp\big(-\frac{3}{2}\tau\big)\sin\big(n\tau\big)\,.
\end{aligned} \tag{2.31c}
$$

where $\Delta x = x^{(i)}$, $\Delta y = y^{(i)} - 1$ and $\Delta\lambda = \lambda^{(i)} + 2$ are the initial conditions.

Using these solutions and the acceleration equation (2.24) one is able to obtain the Hubble's function in the vicinity of the critical point of the focus type. Up to linear terms in initial conditions we have

$$
\begin{aligned}
\left(\frac{H(a)}{H(a_0)}\right)^2 \approx {} & \Omega_{\Lambda,0} + \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3(1+w_m)} + \\
& + \left(\frac{a}{a_0}\right)^{-3/2}\left(\Omega_{cos,0}\cos\left(n\ln\left(\frac{a}{a_0}\right)\right) + \Omega_{sin,0}\sin\left(n\ln\left(\frac{a}{a_0}\right)\right)\right)
\end{aligned} \tag{2.32}
$$

where

$$\Omega_{\Lambda,0} = 1 - \Omega_{M,0} - \Omega_{cos,0} \tag{2.33}$$

and

$$\Omega_{M,0} = \left(1 - \frac{4(4n^2+9)(1-3w_m)(4+3w_m)}{3\delta(4n^2+9(1+2w_m)^2)}\right)\Omega_{m,0}\,, \tag{2.34a}$$

$$\begin{aligned}
\Omega_{cos,0} =& \frac{1}{3\delta}\left(\frac{4(4n^2+9)(1-3w_m)(4+3w_m)}{4n^2+9(1+2w_m)^2}\Omega_{m,i} - \right. \\
& \left. - 3\delta\,\Delta x + 8\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-3/2}\cos\left(n\,\ln\left(\frac{a_0}{a^{(i)}}\right)\right) + \\
& + \frac{1}{6\delta n}\left(\frac{2(4n^2+9)(1-3w_m)(4n^2-15(1+2w_m))}{4n^2+9(1+2w_m)^2}\Omega_{m,i} + \right. \\
& \left. + 15\delta\,\Delta x + 4(4n^2+15)\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-3/2}\sin\left(n\,\ln\left(\frac{a_0}{a^{(i)}}\right)\right)\,,
\end{aligned} \tag{2.34b}$$

$$\begin{aligned}
\Omega_{sin,0} =& \frac{1}{6\delta n}\left(\frac{2(4n^2+9)(1-3w_m)(4n^2-15(1+2w_m))}{4n^2+9(1+2w_m)^2}\Omega_{m,i} + \right. \\
& \left. + 15\delta\,\Delta x + 4(4n^2+15)\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-3/2}\cos\left(n\,\ln\left(\frac{a_0}{a^{(i)}}\right)\right) - \\
& - \frac{1}{3\delta}\left(\frac{4(4n^2+9)(1-3w_m)(4+3w_m)}{4n^2+9(1+2w_m)^2}\Omega_{m,i} - \right. \\
& \left. - 3\delta\,\Delta x + 8\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-3/2}\sin\left(n\,\ln\left(\frac{a_0}{a^{(i)}}\right)\right)\,.
\end{aligned} \tag{2.34c}$$

In this section we obtained two forms of the Hubble's functions for two different types of behaviour in the vicinity of the critical point representing the deSitter state.

2.1 Dust matter

For dust matter and special initial conditions:

$$\Delta x = \frac{4}{\delta}\Omega_{m,i}\,, \qquad \Delta\lambda = -\frac{1}{2}\Omega_{m,i} \tag{2.35}$$

both forms of the Hubble's functions (2.25) and (2.32) take the same form. Namely, in the case of the monotonic approach to the deSitter state (2.25) we have $\Omega_{n,0} = 0$ and $\Omega_{3n,0} = 0$, and in the case of the oscillatory approach (2.32) we have $\Omega_{cos,0} = 0$ and $\Omega_{sin,0} = 0$ and the resulting form of the Hubble's function is

$$\left(\frac{H(a)}{H(a_0)}\right)^2 \approx 1 - \Omega_{M,0} + \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3} \tag{2.36}$$

where

$$\Omega_{M,0} = \left(1 - \frac{16}{3\delta}\right)\Omega_{m,0}\,. \tag{2.37}$$

If we assume that in the model we include only the baryonic matter $\Omega_{m,0} = \Omega_{bm,0}$, then from the recent astronomical observations of the Planck satellite [1] we have that the present total matter density parameter is $\Omega_{M,0} \approx 0.315$ and that the baryonic matter

density parameter is $\Omega_{bm,0} \approx 0.049$. This gives us an opportunity to find the value of the δ parameter of the model, which gives us information about the second derivative of the potential function at the critical point. After a little algebra one finds

$$\delta = \frac{16}{3\left(1 - \frac{\Omega_{M,0}}{\Omega_{bm,0}}\right)} \approx -0.9825\,. \tag{2.38}$$

Next, from the definition of the δ parameter one can directly calculate the value of the second derivative of the scalar field potential at the critical point

$$\Gamma\big|_* = \frac{V''(\phi)V(\phi)}{V'(\phi)^2}\bigg|_* \approx 0.5921\,, \tag{2.39}$$

which indicates that at the critical point the second derivative of the potential function is positive $V''(\phi)\big|_* > 0$.

Then from the assumed initial conditions (2.35) one gets the present values of the phase space variables

$$x(a_0) = \frac{\dot{\phi}}{H\phi}\bigg|_0 \approx -0.1995\,, \qquad \lambda(a_0) = -\phi\frac{V'(\phi)}{V(\phi)}\bigg|_0 \approx -2.0245\,. \tag{2.40}$$

In the Brans-Dicke theory the field ϕ is identified as the inverse of the gravitational coupling which now gives

$$\frac{\dot{G}}{HG}\bigg|_0 \approx 0.1995 \tag{2.41}$$

which, taking the present age of the universe $t_0 = 13.817 \times 10^9$yr, gives

$$\frac{\dot{G}}{G}\bigg|_0 \approx 1.44 \times 10^{-11}\frac{1}{\mathrm{yr}}\,, \tag{2.42}$$

and is in good agreement with other observational constrains on the variability of the gravitational coupling constant [10].

One can also calculate the mass of the Brans-Dicke scalar field. In the Jordan frame it is given by [11]

$$m^2 = \frac{2}{3 + 2\omega_{\mathrm{BD}}}\big(\phi V''(\phi) - V'(\phi)\big), \tag{2.43}$$

which in the previously introduced phase space variables becomes

$$m^2 = \frac{6}{3 + 2\omega_{\mathrm{BD}}}H^2 y^2 \lambda\big(1 + \lambda\Gamma(\lambda)\big)\,. \tag{2.44}$$

Direct calculation of the asymptotic value of the scalar field mass at the critical point gives

$$m^2\big|_* = -\frac{9}{4}H_*^2\frac{\delta}{3 + 2\omega_{\mathrm{BD}}} \tag{2.45}$$

where $H_*^2 \approx H_0^2(1-\Omega_{M,0})$ is the asymptotic value of the Hubble's function and H_0 its present value. Inserting the estimated value of the δ parameter and $H_0 \approx 1.5\times 10^{-33}$eV one obtains

$$m\big|_* \approx 1.84\frac{10^{-33}}{\sqrt{3+2\omega_{\text{BD}}}}\,\text{eV}\,, \tag{2.46}$$

which is an asymptotic value of the mass of the Brans-Dicke scalar field at the deSitter state. Such a ultra-light scalar particles are usually postulated to explain dynamic of galaxies [12] and are treated as a bosonic cold dark matter candidates [13, 14]. To obtain mass of the scalar particle of order 10^{-22} eV one needs

$$\omega_{\text{BD}} \approx -\frac{3}{2} + 10^{-22}\,, \tag{2.47}$$

which is very close to the conformal coupling value [5].

Now we show that the ΛCDM model can also emerge without assumption about specific initial conditions but rather as a specific assumptions about the model parameters. For dust matter $w_m = 0$ the equations (2.26) reduce to

$$\Omega_{M,0} = \bigg(1-\frac{16}{3\delta}\bigg)\Omega_{m,0}\,, \tag{2.48a}$$

$$\Omega_{n,0} = \frac{n+1}{3\delta(2n-3)}\bigg(4n\,\Omega_{m,i} - 3\delta\Delta x + 8(n-3)\,\Delta\lambda\bigg)\left(\frac{a_0}{a^{(i)}}\right)^{-n}\,, \tag{2.48b}$$

$$\Omega_{3n,0} = \frac{n-4}{3\delta(2n-3)}\bigg(-4(n-3)\,\Omega_{m,i} - 3\delta\Delta x - 8n\,\Delta\lambda\bigg)\left(\frac{a_0}{a^{(i)}}\right)^{-3+n}\,, \tag{2.48c}$$

Expanding these quantities for $|n| \ll 1$ and assuming that $n^2 \approx 0$ and $n\Delta x = n\Delta y = n\Delta\lambda \approx 0$ one obtains

$$\Omega_{n,0} \approx \frac{1}{3\delta}\bigg(\delta\Delta x + 8\Delta\lambda\bigg)\,, \tag{2.49a}$$

$$\Omega_{3n,0} \approx \frac{4}{9\delta}\bigg(12\Omega_{m,i} - 3\delta\Delta x\bigg)\left(\frac{a_0}{a^{(i)}}\right)^{-3}\,, \tag{2.49b}$$

and the Hubble's function (2.25) is

$$\left(\frac{H(a)}{H(a_0)}\right)^2 \approx 1 - \left(\Omega_{m,0} - \frac{4}{3}\Delta x\left(\frac{a_0}{a^{(i)}}\right)^{-3}\right) + \left(\Omega_{m,0} - \frac{4}{3}\Delta x\left(\frac{a_0}{a^{(i)}}\right)^{-3}\right)\left(\frac{a}{a_0}\right)^{-3}\,. \tag{2.50}$$

Note that in the general case of the monotonic approach to the deSitter state we have

$$\frac{\delta}{3+2\omega_{\text{BD}}} = \frac{4}{9}n(n-3) \tag{2.51}$$

and the assumption that $|n| \ll 1$ should be interpreted as smallness of the ratio

$$\frac{\delta}{3+2\omega_{\text{BD}}} \approx -\frac{4}{3}n\,. \tag{2.52}$$

From (2.22b) we have $x(a) \approx \Delta x \left(\frac{a}{a^{(i)}}\right)^{-3}$ which gives $x(a_0) \approx \Delta x \left(\frac{a_0}{a^{(i)}}\right)^{-3}$. From the definition of the variable x we obtain that at the present epoch

$$x(a_0) = \frac{\dot{\phi}}{H\phi}\Big|_0. \tag{2.53}$$

If we assume that in the model we include only the baryonic matter $\Omega_m = \Omega_{bm}$ and note that in the Brans-Dicke theory the field ϕ can be identified as the inverse of the gravitational coupling

$$\phi = \frac{1}{G} \tag{2.54}$$

which is now the function of the spacetime location we obtain the following form of the Hubble's function

$$\left(\frac{H(a)}{H(a_0)}\right)^2 \approx 1 - \Big(\Omega_{bm,0} + \Omega_{dm,0}\Big) + \Big(\Omega_{bm,0} + \Omega_{dm,0}\Big)\left(\frac{a}{a_0}\right)^{-3} \tag{2.55}$$

where the present dark matter density parameter is

$$\Omega_{dm,0} = \frac{4}{3}\frac{\dot{G}}{HG}\Big|_0. \tag{2.56}$$

From the Planck satellite data [1] we have that the present value of the density parameter of the dark matter $\Omega_{dm,0} \approx 0.266$ and we obtain

$$\frac{\dot{G}}{HG}\Big|_0 \approx 0.1995\,. \tag{2.57}$$

Two different assumptions lead to very similar results.

2.2 Low-energy string theory limit

The Lagrangian density of the low-energy limit of the bosonic string theory [15–17] can be presented in the following form

$$\mathcal{L} = e^{-2\Phi}\big(R + 4\nabla^\alpha\Phi\,\nabla_\alpha\Phi - \Lambda\big) \tag{2.58}$$

where Φ is the dilaton field. Making the substitution $\phi = e^{-2\Phi}$ one obtains the Brans-Dicke theory with $\omega_{\text{BD}} = -1$ and $V(\phi) = \Lambda\phi$. Neglecting the matter, the two theories are identical, but they differ in their couplings of the scalar field to the other matter [18].

The constant value of the Brans-Dicke parameter $\omega_{\text{BD}} = -1$ leads to

$$\delta = \frac{4}{9}n(n-3) \tag{2.59}$$

and for dust matter $w_m = 0$ equations (2.26) reduce to

$$\Omega_{M,0} = \left(1 - \frac{12}{n(n-3)}\right)\Omega_{m,0}\,, \tag{2.60a}$$

$$\Omega_{n,0} = \frac{n+1}{n(n-3)(2n-3)}\left(3n\,\Omega_{m,i} - n(n-3)\Delta x + 6(n-3)\,\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-n}\,, \tag{2.60b}$$

$$\Omega_{3n,0} = \frac{n-4}{n(n-3)(2n-3)}\left(-3(n-3)\,\Omega_{m,i} - n(n-3)\Delta x - 6n\,\Delta\lambda\right)\left(\frac{a_0}{a^{(i)}}\right)^{-3+n}\,. \tag{2.60c}$$

In this case, we also expand these quantities for $|n| \ll 1$ which indicates that $|\delta| \ll 1$ and we are very close to the quadratic potential function. The resulting Hubble's function up to linear terms in initial condition is

$$\left(\frac{H(a)}{H(a_0)}\right)^2 \approx 1 - \frac{2}{3}\left(\Omega_{m,0} - 2x(a_0) + 4\Delta\lambda\right) + \frac{2}{3}\left(\Omega_{m,0} - 2x(a_0) + 4\Delta\lambda\right)\left(\frac{a}{a_0}\right)^{-3} + \\ + \left(2\Delta\lambda - 4\Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3}\right)\ln\left(\frac{a}{a_0}\right) \tag{2.61}$$

One notices that in the case of the low-energy string theory limit we obtain a different form of the Hubble's function, applying the same type of expansion, but the difference lies in the direct connection of the n parameter with the shape of the potential function at the critical point.

The resulting Hubble's function differs by the presence of the term proportional to the natural logarithm of the scale factor. At the present epoch this term vanishes but in the past it could play critical role.

3 Conclusions

Investigating the emergence of the Standard Cosmological Model ΛCDM from the Brans-Dicke theory of gravity shows the significant problem in modern cosmology, namely, the problem of initial conditions. Cosmology is not a usual physics of the universe because of the presence of this problem at the very beginning. The strategy of the modern cosmology is to study all the possible evolutions for all admissible initial conditions and then test them against the astronomical data. Which evolutional scenario is realised for our universe can be decided by observations and the quality of obtained astronomical data is crucial.

The dynamical systems methods are especially useful in this context. These methods give the opportunity to study all the evolutional paths for all admissible initial conditions. However cosmology is not just pure mathematics because we need to constrain possible evolutions represented by the trajectories in the phase space by the observations, that is we need to find the initial conditions. Cosmology is the science

of the evolution of the universe as well as the initial conditions which give rise to this evolution.

In this paper we have shown how the Standard Cosmological Model (ΛCDM) emerges from the Brans-Dicke theory. There are two ways of approaching the ΛCDM model : a monotonic one and an oscillating one. Which specific case takes place crucially depends on the initial conditions for the universe as well as the model parameters. While the leading term in both cases represents ΛCDM type evolution, there are additional terms which describe the asymptotic state of the system (a node or a focus type critical point in the phase space). The density parameters for the dark energy as well as for the dark matter are emergent parameters from the Brans-Dicke cosmology. The values of these parameters are fragile due to value and the sign of the second derivative of the scalar field potential at the critical point.

Finally, from the recent Planck satellite data, we obtain constraint on the variability of the gravitational coupling constant as well as the lower limit of the mass of the Brans-Dicke scalar field at the deSitter state.

Our results are independent on the special form of the potential function for the Brans-Dicke scalar field and indicate that the value of its second derivative at the critical point is crucial for the expansion scenario of current universe.

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